3

Thin Walled Pressure Vessels

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§3.1. Introduction

This Lecture continues with the theme of the last one: using average stresses instead of point stresses to quickly get results useful in preliminary design or in component design. We look at more complicated structural configurations: thin wall pressure vessels, which despite their apparently higher complexity can be treated directly by statics if both geometry and loading are sufficiently simple.

The main difference with respect to the component configurations treated in the previous lecture is that the state of stress in the vessel wall is *two dimensional*. More specifically: plane stress. As such they will provide examples for 2D stress-displacement analysis once 2D strains and multidimensional material laws are introduced in Lectures 4–5.

§3.2. Pressure Vessels

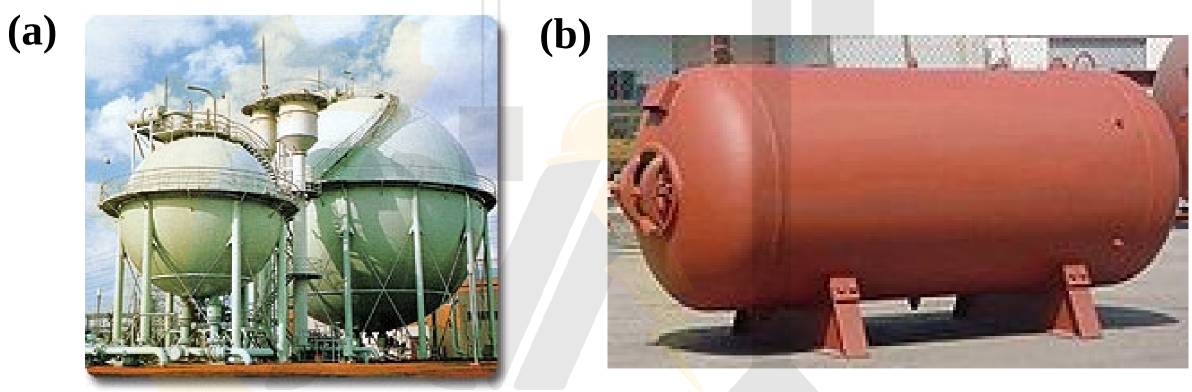

FIGURE 3.1. Pressure vessels used for fluid storage: (a) spherical tanks, (b) cylindrical tank.

Thin wall pressure vessels (TWPV) are widely used in industry for storage and transportation of liquids and gases when configured as tanks. See Figure 3.1. They also appear as components of aerospace and marine vehicles such as rocket and balloon skins and submarine hulls (although in the latter case the vessel is externally pressurized, violating one of the assumptions listed below). Two geometries will be examined in this lecture:

- Cylindrical pressure vessels.
- Spherical pressure vessels.

The walls of an *ideal* thin-wall pressure vessel act as a membrane (that is, they are unaffected by bending stresses over most of their extent). A sphere is the optimal geometry for a closed pressure vessel in the sense of being the most structurally efficient shape. A cylindrical vessel is somewhat less efficient for two reasons: (1) the wall stresses vary with direction, (2) closure by end caps can alter significantly the ideal membrane state, requiring additional local reinforcements. However the cylindrical shape may be more convenient to fabricate and transport.

§3.3. Assumptions

The key assumptions used here are: wall thinness and geometric symmetries. These make possible to obtain average wall stresses analysis with simple free-body diagrams (FBD). Here is a more detailed list of assumptions:

1. *Wall Thinness*. The wall is assumed to be *very thin* compared to the other dimensions of the vessel. If the thickness is t and a characteristic dimension is R (for example, the radius of the cylinder or sphere) we assume that

$$t/R << 1, \quad \text{or} \quad R/t >> 1 \tag{3.1}$$

   Usually $R/t > 10$. As a result, we may assume that the stresses are *uniform* across the wall.

2. *Symmetries*. In cylindrical vessels, the geometry and the loading are cylindrically symmetric. Consequently the stresses may be assumed to be independent of the angular coordinate of the cylindrically coordinate system. In spherical vessels, the geometry and the loading are spherically symmetric. Therefore the stresses may be assumed to be independent of the two angular coordinates of the spherical coordinate system and in fact are the same in all directions.

3. *Uniform Internal Pressure*. The internal pressure, denoted by p, is uniform and everywhere positive. If the vessel is also externally pressurized, for example subject to athmospheric pressure, p is defined by subtracting the external pressure from the internal one, a difference called *gage pressure*. If the external pressure is higher, as in the case of a submarine hull, the stress formulas should be applied with extreme caution because another failure mode: instability due to wall buckling, may come into play. See Section 3.5.

4. *Ignoring End Effects*. Features that may affect the symmetry assumptions are ignored. This includes supports and cylinder end caps. The assumption is that disturbances of the basic stress state are confined to local regions and may be ignored in basic design decision such as picking up the thickness away from such regions.

We study the two simplest geometries next.

§3.4. Cylindrical Vessels

We consider a cylindrical vessel of radius R, thickness t loaded by internal pressure p. We use the cylindrical coordinate system (x, r, θ) depeicted in Figure 3.2(a), in which

- x axial coordinate
- θ angular coordinate, positive as shown
- r radial coordinate

§3.4.1. Stress Assumptions

Cut the cylinder by two normal planes at x and $x + dx$, and then by two planes θ and $\theta + d\theta$ as shown in Figure 3.2(a). The resulting material element, shown in exploded view in Figure 3.2(b) has six surfaces. The outer surface $r = R$ is stress free. Thus

$$\sigma_{rr} = \tau_{rx} = \tau_{r\theta} = 0 \quad \text{at} \quad r = R \tag{3.2}$$

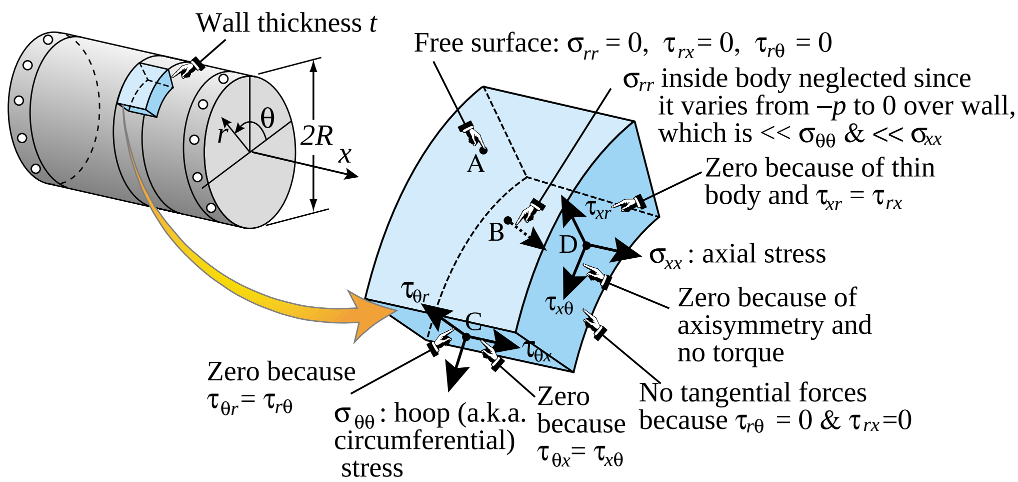

FIGURE 3.2. Wall material element of a pressurized cylindrical vessel referred to cylindrical coordinates. Note that thickness is grossly exaggerated for visibility.

On the inner surface $r = R - t$ there is a compressive normal stress that balances the applied pressure but no tangential stresses. Thus

$$\sigma_{rr} = -p, \quad \tau_{rx} = \tau_{r\theta} = 0 \quad \text{at} \quad r = R - t \tag{3.3}$$

Since the wall is thin, we can confidently assume that

$$\tau_{rx} = \tau_{r\theta} = 0 \quad \text{for all } r \in [R - t, R] \tag{3.4}$$

whereas σ_{rr} varies from $-p$ to zero. Later on we will find that σ_{rr} is much smaller than the other two normal stresses, and in fact may be neglected (set to zero). Because $\tau_{rx} = \tau_{zr}$ and $\tau_{r\theta} = \tau_{\theta r}$ on account of shear stress reciprocity, we conclude that

$$\tau_{zr} = \tau_{\theta r} = 0 \quad \text{for all } r \text{ inside wall} \tag{3.5}$$

The normal stresses σ_{xx} and σ_{zz} are called *axial stress* and *circumferential* or *hoop stress*, respectively. The last wall stress component is $\tau_{\theta x} = \tau_{x\theta}$, which is the wall shear stress. Because of symmetry assumptions on the geometry and loading (no torque), this stress is zero. These stress assumptions are graphically displayed, with annotations, in Figure 3.2.

Displaying the wall stress state using the stress matrix and taking the axes in order $\{x, \theta, r\}$ for convenience, we have

$$\begin{bmatrix} \sigma_{xx} & \tau_{x\theta} & \tau_{xr} \\ \tau_{\theta x} & \sigma_{\theta\theta} & \tau_{\theta r} \\ \tau_{rx} & \tau_{r\theta} & \sigma_{rr} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & 0 & 0 \\ 0 & \sigma_{\theta\theta} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.6}$$

Comparing this to the 2D stress state introduced in Lecture 1, we observe that the cylinder vessel wall is in *plane stress*.

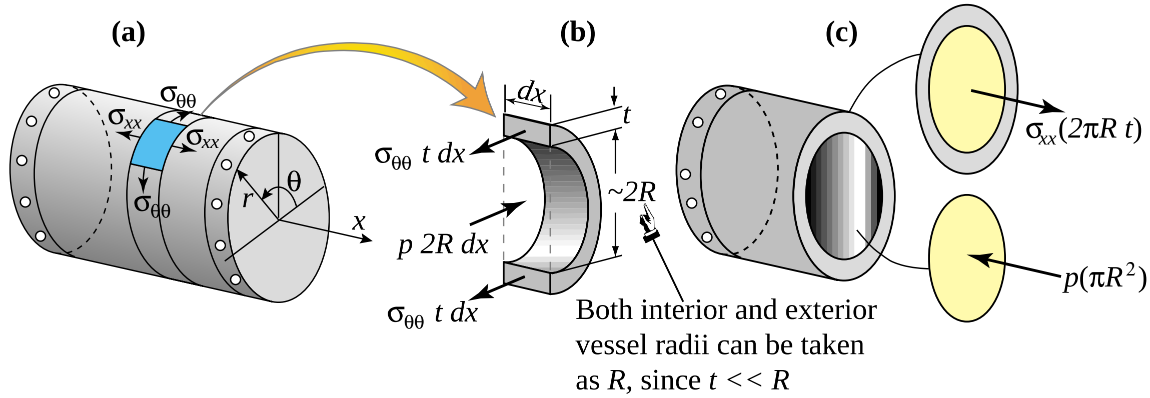

FIGURE 3.3. Free body diagrams (FBD) to get the averaged hoop and longitinal wall stresses in a pressurized thin-wall cylindrical vessel

§3.4.2. Free Body Diagrams

The nonzero normal stress components σ_{xx} and $\sigma_{\theta\theta}$ in (3.6) act as sketched in Figure 3.3(a). Both components are *uniform* across the wall thickness and throughout the vessel (excluding possible end effects, which are discussed in Section 3.5). To find their value in terms of the data: p, R, and t, we use the two FBD drawn in Figure 3.3(b,c). Details will be worked out in class. The result is

$$\boxed{\sigma_{\theta\theta} = \frac{p\,R}{t}, \quad \sigma_{xx} = \frac{p\,R}{2t} = \tfrac{1}{2}\,\sigma_{\theta\theta}.} \tag{3.7}$$

Neither stress depends on position. Because the hoop stress is twice the axial stress, it will be the controlling one in a strength design. For example if $R/t = 100$, which is a typical vessel thickness in aerospace applications, then $\sigma_{\theta\theta} = 100\,p$ and $\sigma_{xx} = 50\,p$. Since σ_{rr} is of the order of p as previously discussed, it follows that neglecting it is justified.

We can summarize our findings by showing the stress matrix now expressed in terms of the data:

$$\begin{bmatrix} \sigma_{xx} & 0 & 0 \\ 0 & \sigma_{\theta\theta} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \frac{p\,R}{2t} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.8}$$

§3.5. Spherical Pressure Vessel

A similar approach can be used to derived an expression for an internally pressurized thin-wall spherical vessel. We use spherical coordinates $\{r, \theta, \phi\}$, as illustrated in Figure 3.4(a).

§3.5.1. Stress Assumptions

Reasoning as in the preceding case, we find that

1. All shear stresses are zero: $\tau_{r\phi} = \tau_{\phi r} = 0$, $\tau_{r\theta} = \tau_{\theta r} = 0$ and $\tau_{\theta\phi} = \tau_{\phi\theta} = 0$.
2. The normal stress σ_{rr} varies from zero on the outside free surface to the negative of the pressure p on the inside surface. Again we will neglect this value when compared to the other normal stresses and justify this assumption *a posteriori*.

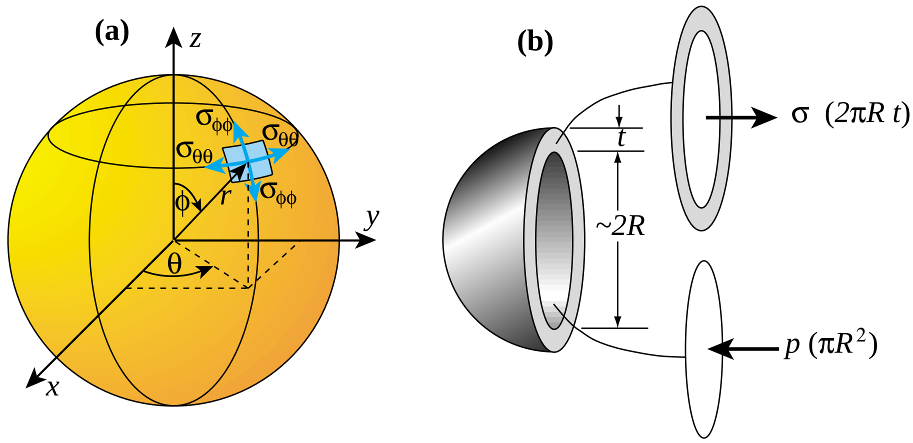

FIGURE 3.4. Stress analysis of a spherical pressure vessel in spherical coordinates. Once again thickness is grossly exaggerated for visibility.

3. The normal stresses $\sigma_{\theta\theta}$ and $\sigma_{\phi\phi}$ are equal and constant over the entire vessel. For simplicity we will use the abbreviation $\sigma = \sigma_{\theta\theta} = \sigma_{\phi\phi}$.

For convenience in writing out the stress matrix we will order the axes as $\{\theta, \phi, r\}$. As per the preceding discussion, the stresses at any wall point have the configuration

$$\begin{bmatrix} \sigma_{\theta\theta} & \tau_{\theta\phi} & \tau_{\theta r} \\ \tau_{\phi\theta} & \sigma_{\phi\phi} & \tau_{\phi r} \\ \tau_{r\theta} & \tau_{r\phi} & \sigma_{rr} \end{bmatrix} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.9}$$

This shows again that the vessel wall is in a *plane stress* state.

§3.5.2. Free Body Diagram

To find σ we cut the sphere into two hemispheres as shown in Figure 3.4(b). The FBD gives the equilibrium condition $\sigma\, 2\pi\, Rt = p\, \pi\, R^2$, whence

$$\boxed{\sigma = \frac{p\,R}{2t}} \tag{3.10}$$

Any section that passes through the center of the sphere yields the same result.

We can summarize our findings by showing the stress matrix expressed in terms of the original data:

$$\begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{bmatrix} = \frac{p\,R}{2t} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.11}$$

Comparing to (3.9) shows that for the same p, R and t the spherical geometry is twice as efficient in terms of wall stress. Why? This is explained in the next section.

§3.6. Remarks on Pressure Vessel Design

For comparable radius, wall thickness and internal pressure the maximum normal stress in a spherical pressure vessel is one half as large as that in a cylindrical one. The reason can be understood by comparing Figures 3.5(a,b). In the cylindrical vessel the internal pressure is resisted by the hoop stress in “arch action” whereas the axial stress does not contribute. In the spherical vessel the double curvature means that all stress directions around the pressure point contribute to resisting the pressure. The cylindrical geometry, however, can result in more efficient assignment of container space as well as stacking and better aerodynamics: a spherical rocket does not look quite right.

One important point for designers is: what happens at the ends of a cylindrical vessel? Suppose for instance that a cylinder is closed by hemispherical end caps, as pictured in Figure 3.6(a). If the cylinder and the end caps were allowed to deform independently of each other under pressurization they would tend to expand as indicated by the dashed lined in that figure. The cylinder and the ends would in general expand by different amounts. But since physical continuity of the wall must be maintained, the necessary adjustment in the displacement would produce local bending as well as shear stresses in the vicinity of the juncture, as pictured in Figure 3.6(b). If thick plates are used instead of relatively flexible hemispherical ends, those juncture stresses would increase considerably as shown in Figure Figure 3.6(b). For this reason, the ends of cylindrical pressure vessels must be designed carefully, and flat ends are should be avoided if possible.

Most pressure vessels are fabricated from curved metal sheets that are joined by welds. Two weld types: double dillet lap joint and double welded butt joint with V grooves are shown in Figures 3.7(a) and (b), respectively. Of these preference should be given to the latter as it avoids across-the-weld load transmission eccentricity.

It should be emphasized that the formulas derived for TWPV in this Lecture should be used only for cases of *internal pressure*. (Or, more precisely, the internal pressure exceeds the external one). If a vessel is to be designed for *external pressure*, as in the case of a submarine or vacuum tank, *wall buckling*, whether elastic or inelastic, may well become the critical failure mode. Should that be the case, the previous wall stress formulas are only part of the design.

§3.7. Numerical Example

§3.7.1. CExample: Cylindrical Tank With Bolted Lids

This is Example 4.18 of Vable’s book. It is reproduced here as it combines the results of this Lecture with the bolt-design-by-average-stress technique described in Lecture 2.

Each lid is bolted to the tank of Figure 3.8(a) along the flanges using 1-in-diameter bolts. The tank is made from sheet metal that is $\frac{1}{2}$ in thick and can sustain a maximum hoop stress of 24 ksi in tension. The normal stress in the bolts is to be limited to 60 ksi in tension. A manufacturer can make tanks of diameters varying from 2 ft through 8 ft in steps of 1 ft. Develop a table that the manufacturer can use to advise custometrs of the size of the tank and the number of bolts per lid needed to hold a desired gas pressure.

Solution. The area of each bolt is $A_{bolt} = \frac{1}{4}\pi(1\,\text{in})^2 = \frac{1}{4}\pi$ sq in. From the hoop stress equation $\sigma_{\theta\theta} = pR/t$ we get

$$\sigma_{\theta\theta} = \frac{p\,R}{\frac{1}{2}\,\text{in}} \leq 24\,\text{ksi} = 24{,}000\,\text{psi}, \quad \text{whence} \quad p \leq \frac{12{,}000}{R}\,\text{psi} \tag{3.12}$$

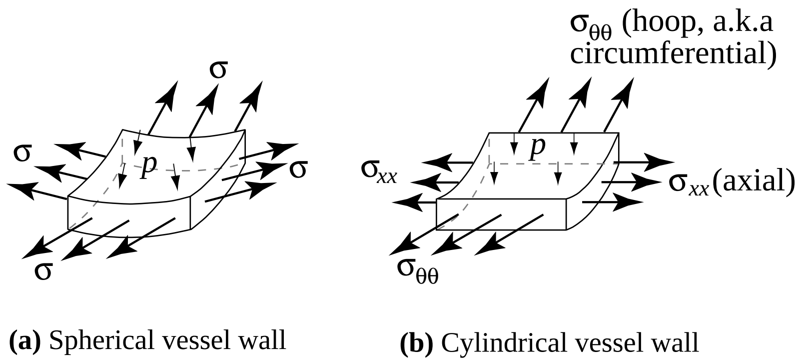

FIGURE 3.5. Why spherical vessels are more structurally efficient.

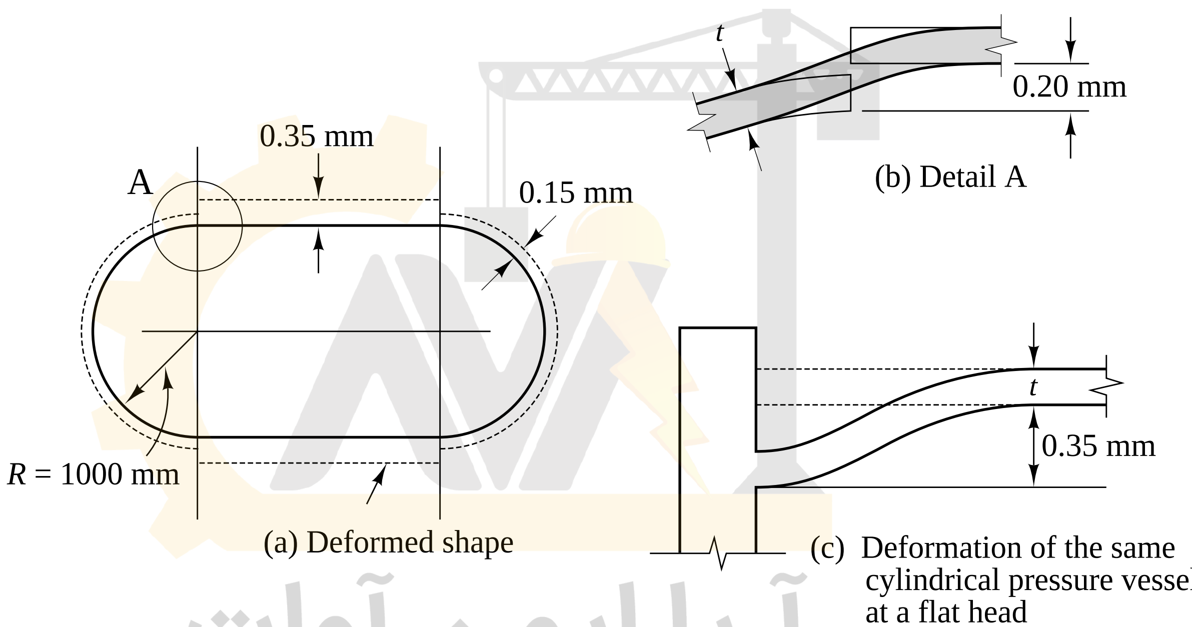

FIGURE 3.6. End effects in cylindrical vessels.

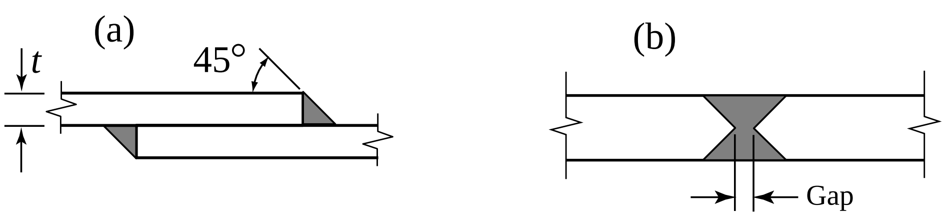

FIGURE 3.7. Welds in pressure vessels: two configurations.

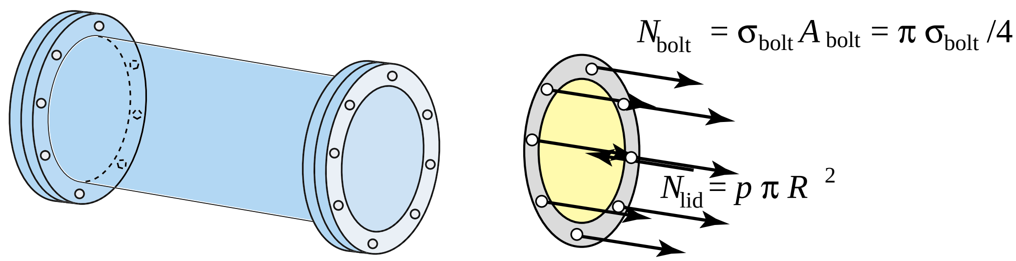

FIGURE 3.8. Cylindrical tank with bolted lids for Example 3.2.

The FBD of the lid is shown in Figure 3.8(b). Force equilibrium in the x direction gives

$$n N_{bolt} = N_{lid}, \quad \sigma_{bolt} = \frac{4 p R^2}{n} \leq 60{,}000. \tag{3.13}$$

Substituting for p gives

$$\frac{4 \times 12{,}000 R}{n} \leq 60{,}000, \quad \text{or} \quad n \geq 0.8 R. \tag{3.14}$$

Rewriting these inequalities in terms of the diameter $D = 2R$ of the tank we get

$$p \leq \frac{24{,}000}{D}, \quad \text{and} \quad n \geq 0.4D, \qquad D \text{ in inches.} \tag{3.15}$$

We now tabulate the maximum pressure p and the number of bolts n in terms of D as we step from $D = 2$ ft $= 24$ in through $D = 8$ ft $= 96$ in. The values of p are rounded up to the nearest integer multiple of 5 whereas values of n are reported by rounding up to the nearest integer.

Table 3.1. Results for Example 4.18 of Vable

Tank Diameter (ft)	Max Pressure (psi)	Min # of Bolts
2	1000	10
3	665	15
4	500	20
5	400	24
6	330	30
7	280	34
8	250	39